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Actions on positively curved manifolds and boundary in the orbit space

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We study isometric actions of compact Lie groups on complete orientable positively curved n-manifolds whose orbit spaces have non-empty boundary in the sense of Alexandrov geometry. In particular, we classify quotients of the unit sphere by actions of compact simple Lie groups with non-empty boundary. We deduce from this the list of representations of compact simple Lie groups that admit non-trivial reductions. As a tool of special interest, we introduce a new geometric invariant of a compact symmetric space, namely, the minimal number of points in a "spanning set" of the space.

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1. Introduction

1.1. General observations

For an isometric action of a compact Lie group G on a complete Riemannian manifold M with orbit space $X = M/G$ stratified by orbit types, the boundary of X consists of the most important singular strata of X; here the *boundary* ∂X is defined as the closure of the union of all strata of codimension one of X. In case M is positively curved, this notion of boundary coincides with the boundary of X as an Alexandrov space and has a bearing on the geometry and topology of X. For instance it is easy to see that ∂X is non-empty if and only if X is contractible (for the 'only if' part one uses the fact that the distance to the boundary is a strictly concave function hence admits a unique point of maximum, a "soul point"; the 'if' part follows from the fact the Alexander-Spanier $\mathbb{Z}_2$-cohomology in top degree of X is non-trivial if $\partial X = \varnothing$ [15, Lemma 1]). In general, the boundary plays an important role in some proofs in the literature; see e.g. main results in [32], or [2, Theorem 1.4] and [12, §5.3].

1.2. The case of quotients of the sphere

It follows from the slice theorem that the presence of boundary is a local condition, in the sense that $X = M/G$ has non-empty boundary if and only if there exists a point $p \in M$ such that the slice representation of the isotropy group G_p on the normal space $\nu_p(Gp)$ to the orbit Gp has orbit space with non-empty boundary. The orbit space of an orthogonal representation is a metric cone over the orbit space of the corresponding unit sphere, so also the boundary of the former is a metric cone over the boundary of the latter. These remarks show that the special case of quotients of the unit sphere with non-empty boundary plays a distinguished role.

In fact, as a main consequence of our methods, we deduce a rather simple criterion for the existence of boundary for quotients of spheres (or more generally, positively curved manifolds) by *simple* groups.

Theorem 1.1. *Let G be a compact connected simple Lie group. Then there is an explicit, positive integer $\mathcal{L}_G$, depending only on the local isomorphism class of G, such that: For every effective and isometric action of G on a connected complete orientable Riemannian manifold M of positive sectional curvature, if $\dim M \geq \mathcal{L}_G$ and the orbit-space has non-empty boundary, then the G-fixed point set $M^G \neq \varnothing$ and $\dim M^G \geq \dim M - \mathcal{L}_G$.*

The number $\mathcal{L}_G$ is easy to determine (cf. Table 4 for its values) and has geometric meaning, namely

$$\mathcal{L}_G := \max_K \{\ell_{G/K}(4 + \dim G/K)\}, \tag{1.1}$$

where K runs through all symmetric subgroups of G with maximal rank, and $\ell_{G/K}$ is defined as the minimum number ℓ such that there exist ℓ points in G/K not contained in any proper closed connected totally geodesic submanifold (cf. section 4).

The number $\ell_{G/K}$ is a natural, geometric invariant of a compact symmetric space G/K, which is related to the minimum number of involutions of G necessary to topologically generate the group (Proposition 4.2), and to the minimum number of generic points of G/K which are not simultaneously fixed by a non-identity isometry in G (Proposition 4.3). In a sense, it is the minimum number of points "spanning" G/K, and loosely alludes to the concept of linearly independence in Linear Algebra. For instance, for the sphere we have $\ell_{S^n} = n+1$. However, the case of rank one symmetric spaces and Grassmannians turns out to be special, as $\ell_{G/K} = 3$ for the other spaces that we compute (cf. Theorem 4.6).

Applying Theorem 1.1 to orthogonal actions on unit spheres yields that a representation of a compact connected simple Lie group G on an Euclidean space V that has no trivial components can have orbit space with non-empty boundary only if $\dim V \leq \mathcal{L}_G$. We obtain a classification of such representations by combining this remark with a result about reducible representations (Corollary 7.3).

Theorem 1.2. *The representations V of compact connected simple Lie groups G with non-empty boundary in the orbit space are listed in Tables 1 and 2, up to a trivial component and up to an outer automorphism. In the irreducible case (Table 1), we also indicate the kernel of the representation in those cases in which it is non-trivial, the effective principal isotropy group, and whether the representation is polar, toric or quaternion-toric (we recall these concepts in subsection 1.3).*

To exemplify the usefulness of the remark about the existence of boundary being a local property, we give the following result. The special thing about the groups listed in the statement of the next corollary is that according to Theorem 1.2 they are simple Lie groups for which a given representation has non-empty boundary in the orbit space if and only if it is polar.

Corollary 1.3. *Let G be one of the following simple Lie groups:*

$$\mathsf{SU}(2),\ \mathsf{SU}(n)/\mathbb{Z}_n\ (n \geq 3),\ \mathsf{SU}(8)/\mathbb{Z}_4,\ \mathsf{SO}(n)/\{\pm 1\}\ (n \geq 6\ \text{even}),$$
$$\mathsf{SO}'(16),\ \mathsf{Sp}(n)/\{\pm 1\}\ (n \geq 4),\ \mathsf{E}_6/\mathbb{Z}_3,\ \mathsf{E}_7/\mathbb{Z}_2,\ \mathsf{E}_8$$

*($\mathsf{SO}'(16)$ denotes a group isomorphic to the image of $\mathsf{Spin}(16)$ under a half-spin representation). Consider an **effective** isometric action of G on a connected simply-connected compact Riemannian manifold M of positive sectional curvature and dimension $n > \mathcal{L}_G$ (see Table 4 for the explicit values of $\mathcal{L}_G$). Then the orbit space $X = M/G$ has non-empty boundary if and only if the action is polar; further, in this case M is equivariantly diffeomorphic a compact rank one symmetric space with a linearly induced action.*

Table 1
Irreducible representations of compact simple Lie groups with non-empty boundary in the orbit space.

G	Kernel	V	Property	Effective p.i.g.
$\mathsf{SU}(2)$	—	$\mathbb{C}^2$	polar	1
$\mathsf{SO}(3)$	—	$\mathbb{R}^3$ $\mathrm{S}_0^2\mathbb{R}^3 = \mathbb{R}^5$	polar	T^1 $\mathbb{Z}_2^2$
$\mathsf{SU}(n)$ $(n \geq 3)$	— $\mathbb{Z}_n$ $\{\pm 1\}$ if n is even	$\mathbb{C}^n$ Ad $\mathrm{S}^2\mathbb{C}^n$	polar toric	$\mathsf{SU}(n-1)$ T^{n-1} $\mathbb{Z}_2^{n-1}/\ker$
$\mathsf{SU}(n)$ $(n \geq 5)$	$\{\pm 1\}$ if n is even	$\Lambda^2\mathbb{C}^n$	polar (n odd) toric (n even)	$\mathsf{SU}(2)^{\lfloor \frac{n}{2} \rfloor}/\ker$
$\mathsf{SU}(6)$	—	$\Lambda^3\mathbb{C}^6 = \mathbb{H}^{10}$	q-toric	T^2
$\mathsf{SU}(8)$	$\mathbb{Z}_4$	$[\Lambda^4\mathbb{C}^8]_\mathbb{R}$	polar	$\mathbb{Z}_2^7$
$\mathsf{SO}(n)$ $(n \geq 5)$	— $\{\pm 1\}$ if n is even	$\mathbb{R}^n$ $\Lambda^2\mathbb{R}^n = \mathrm{Ad}$ $\mathrm{S}_0^2\mathbb{R}^n$	polar	$\mathsf{Spin}(n-1)$ $\mathsf{T}^{\lfloor \frac{n}{2} \rfloor}$ $\mathbb{Z}_2^{n-1}/\ker$
$\mathsf{Spin}(7)$	—	$\mathbb{R}^8$ (spin)	polar	G_2
$\mathsf{Spin}(8)$	$\mathbb{Z}_2$	$\mathbb{R}^8_\pm$ (half-spin)	polar	$\mathsf{Spin}'(7)$
$\mathsf{Spin}(9)$	—	$\mathbb{R}^{16}$ (spin)	polar	$\mathsf{Spin}(7)$
$\mathsf{Spin}(10)$	—	$\mathbb{C}^{16}_\pm$ (half-spin)	polar	$\mathsf{SU}(4)$
$\mathsf{Spin}(11)$	—	$\mathbb{H}^{16}$ (spin)	—	1
$\mathsf{Spin}(12)$	$\mathbb{Z}_2$	$\mathbb{H}^{16}_\pm$ (half-spin)	q-toric	$\mathsf{Sp}(1)^3$
$\mathsf{Spin}(16)$	$\mathbb{Z}_2$	$\mathbb{R}^{128}_\pm$ (half-spin)	polar	$\mathbb{Z}_2^8$
$\mathsf{Sp}(n)$ $(n \geq 3)$	— $\{\pm 1\}$	$\mathbb{C}^{2n} = \mathbb{H}^n$ $[\mathrm{S}^2\mathbb{C}^{2n}]_\mathbb{R} = \mathrm{Ad}$ $[\Lambda_0^2\mathbb{C}^{2n}]_\mathbb{R}$	polar	$\mathsf{Sp}(n-1)$ T^n $\mathsf{Sp}(1)^n/\{\pm 1\}$
$\mathsf{Sp}(3)$	—	$\Lambda_0^3\mathbb{C}^6 = \mathbb{H}^7$	q-toric	$\mathbb{Z}_2^2$
$\mathsf{Sp}(4)$	$\{\pm 1\}$	$[\Lambda_0^4\mathbb{C}^8]_\mathbb{R}$	polar	$\mathbb{Z}_2^6$
G_2	—	$\mathbb{R}^7$ Ad	polar	$\mathsf{SU}(3)$ T^2
F_4	—	$\mathbb{R}^{26}$ Ad	polar	$\mathsf{Spin}(8)$ T^4
E_6	—	$\mathbb{C}^{27}$	toric	$\mathsf{Spin}(8)$
E_6	$\mathbb{Z}_3$	Ad	polar	T^6
E_7	—	$\mathbb{H}^{28}$	q-toric	$\mathsf{Spin}(8)$
E_7	$\mathbb{Z}_2$	Ad	polar	T^7
E_8	—	Ad	polar	T^8

1.3. The complexity of orbit spaces

Our results also have a bearing on understanding the “complexity” of quotients of the unit sphere. In the case of orthogonal representations of a compact Lie group on vector spaces (or more generally, isometric actions on positively curved manifolds), the following criteria have been used to describe representations whose geometry is not too complicated, namely:

Table 2
Reducible representations of compact simple Lie groups with non-empty boundary in the orbit space.

$\mathsf{SU}(n)$	$k\,\mathbb{C}^n$ $\mathbb{C}^n \oplus \Lambda^2\mathbb{C}^n$	$2 \le k \le n-1$ $n \ge 5$
$\mathsf{SU}(4)$	$k\,\mathbb{R}^6 \oplus \ell\,\mathbb{C}^4$ $\mathbb{R}^6 \oplus \mathrm{Ad}$	$2 \le k+\ell \le 3$ $-$
$\mathsf{SO}(n)$	$k\,\mathbb{R}^n$ $\mathbb{R}^n \oplus \mathrm{Ad}$	$2 \le k \le n-1$ $n \ge 5$
$\mathsf{Sp}(2)$	$\mathbb{H}^2 \oplus \mathbb{R}^5$	$-$
$\mathsf{Spin}(7)$	$k\,\mathbb{R}^7 \oplus \ell\,\mathbb{R}^8$	$2 \le k+\ell \le 4$
$\mathsf{Spin}(8)$	$k\,\mathbb{R}^8 \oplus \ell\,\mathbb{R}^8_+ \oplus m\,\mathbb{R}^8_-$	$2 \le k+\ell+m \le 5$
$\mathsf{Spin}(9)$	$k\,\mathbb{R}^{16}$ $\mathbb{R}^{16} \oplus k\,\mathbb{R}^9$ $2\mathbb{R}^{16} \oplus k\,\mathbb{R}^9$	$2 \le k \le 3$ $1 \le k \le 4$ $0 \le k \le 2$
$\mathsf{Spin}(10)$	$\mathbb{C}^{16} \oplus k\,\mathbb{R}^{10}$	$1 \le k \le 3$
$\mathsf{Spin}(12)$	$\mathbb{H}^{16} \oplus \mathbb{R}^{12}$	$-$
$\mathsf{Sp}(n)$	$k\,\mathbb{C}^{2n}$ $\mathbb{C}^{2n} \oplus [\Lambda^2_0\mathbb{C}^{2n}]_{\mathbb{R}}$	$2 \le k \le n$ $n \ge 3$
$\mathsf{Sp}(3)$	$2\,[\Lambda^2_0\mathbb{C}^6]_{\mathbb{R}}$	$-$
G_2	$k\,\mathbb{R}^7$	$2 \le k \le 3$
F_4	$2\,\mathbb{R}^{26}$	$-$

In case of $\mathsf{Spin}(8)$, the prime in $\mathsf{Spin}'(7)$ refers to a non-standard $\mathsf{Spin}(7)$-subgroup; in case of $\mathsf{Spin}(n)$, $\mathrm{S}^2_0\mathbb{R}^n = \mathrm{S}^2\mathbb{R}^n \ominus \mathbb{R}$; in case of $\mathsf{Sp}(n)$, $\Lambda^k_0\mathbb{C}^{2n} = \Lambda^k\mathbb{C}^{2n} \ominus \Lambda^{k-2}\mathbb{C}^{2n}$; and $[V]_{\mathbb{R}}$ denotes a real form of V.

(i) The principal isotropy group is non-trivial.
(ii) There exists a non-trivial reduction, that is, a representation of a group with smaller dimension and isometric orbit space.
(iii) The cohomogeneity, or codimension of the principal orbits, is “low”.

It is known that (i) implies (ii) [33], and (ii) implies having non-empty boundary [12, Proposition 5.2]. Indeed in case (i), the number of faces of the boundary of the orbit space of an isometric action on a positively curved manifold controls the number of simple factors and the dimension of the center of the principal isotropy group [35, Corollary 12.1]; here a *face* is defined as the closure of a component of a codimension one stratum. We see a posteriori that to some extent (iii) is also related to having non-empty boundary [19]. Representations with non-trivial principal isotropy group have been partially classified in [18] (however, note that the spin representation of $\mathsf{Spin}(14)$ listed in Table A therein indeed has trivial principal isotropy group; cf. [14, Remark 3.2]), and the systematic study of representations with non-trivial reductions (beyond polar representations) has been initiated in [12].

Recall that a representation is called *polar* if it admits a reduction to a representation of a finite group, and it is called *toric* (resp. *quaternion-toric*) if it is non-polar and it admits a reduction to a representation of a group whose identity component is Abelian (resp. is isomorphic to $\mathsf{Sp}(1)^k$ for some $k > 0$). These classes are mostly related to the isotropy representations of symmetric spaces. Polar representations are classified in [7] (see also [3]). Toric irreducible representations are classified in [13] (see also [30] for some partial results in the reducible case). Quaternion-toric irreducible representations are classified in [9].

As another corollary to Theorem 1.2, we deduce:

Corollary 1.4. *An irreducible representation of a compact connected simple Lie group admits a non-trivial reduction if and only if it is polar, toric or q-toric.*

Up to orbit-equivalence, the representations in Corollary 1.4 also coincide with the representations of compact connected simple Lie groups with non-trivial principal isotropy group [18, ch. I, § 2]. Further, their minimal reductions are obtained from the fixed point set of a principal isotropy group, after possibly enlarging the group to an orbit-equivalent action. The (complexification of the) isometry between the orbit spaces given by this kind of reduction was shown to be an isomorphism of affine algebraic varieties in [25]; in particular, it is a diffeomorphism in the sense of [32]. In this sense, Corollary 1.4 can also be seen as a small step toward proving the conjecture that a version of the Myers-Steenrod theorem holds for orbit spaces, namely, that the smooth structure is determined by the metric structure (see [1, §1.1, 1.2, 1.3] and [2, §1]).

1.4. Quaternionic representations

The following result came out of discussions of the first named author with Ricardo Mendes. It implies that the identity component of the isometry group of the orbit space of an irreducible representation of quaternionic type with cohomogeneity at least two is isomorphic to $\mathsf{Sp}(1)$ or $\mathsf{SO}(3)$ (compare [28]).

Corollary 1.5. *Let $\rho : G \to \mathsf{O}(V)$ be an irreducible representation of quaternionic type of a compact connected Lie group G with cohomogeneity $c(\rho) \geq 2$. Consider the natural enlargement $\hat{\rho} : \hat{G} \to \mathsf{O}(V)$, where $\hat{G} = G \times \mathsf{Sp}(1)$. Then the cohomogeneities of these representations satisfy*

$$c(\rho) = c(\hat{\rho}) + 3.$$

In particular, $\hat{\rho}$ is not orbit-equivalent to ρ.

1.5. Dimension estimate

After a presentation of our applications, we have now come to the rather technical statement of our most general main result, although in the present paper we have not had the opportunity of applying it in its full force. It is a general estimate on the dimension of a positively curved manifold on which a Lie group acts with orbit space with non-empty boundary. The normal subgroup N in Theorem 1.6 contains all the information about the boundary of X and has a fixed point; its existence is an act of balance between condition (a) that restricts the largeness of N, and condition (c) that restricts its smallness. Note that in case G is simple, the theorem is just saying that G has a fixed point.

Theorem 1.6. *Let G be a compact connected Lie group acting isometrically and effectively on a connected complete orientable n-manifold M of positive sectional curvature. Assume that $X = M/G$ has non-empty boundary and*

$$n > \alpha_G + \beta_G \tag{1.2}$$

where

$$\alpha_G = 2\dim G_{ss} + 8\,\mathrm{rk}\,G_{ss} + 4\,\mathrm{nsf}\,G_{ss} \quad \textit{and} \quad \beta_G = 2\dim Z(G);$$

here $Z(G)$ denotes the center of G, $G_{ss} = G/Z(G)$ its semisimple part and $\mathrm{nsf}()$ refers to the number of simple factors of a semisimple group. Then there exists a positive-dimensional normal subgroup N of G such that:

(*a*) *The fixed point set M^N is non-empty (and G-invariant); let B be a component of M^N containing principal G-orbits.*
(*b*) *B/G has empty boundary and is contained in all faces of X.*
(*c*) *In particular:*
  - (i) *N contains, up to conjugation, all isotropy groups of G corresponding to orbit types of strata of codimension one in X.*
  - (ii) *At a generic point of B, the slice representation of N has orbit space with non-empty boundary.*
  - (iii) *If, in addition, M is simply-connected, then the statement in (ii) is true with N replaced by its identity component N^0.*

This theorem will be proved in section 6. A rather straightforward modification of the argument proves a strengthened version in which M is only assumed to have positive k-th Ricci curvature, inequality (1.2) is assumed to hold with n replaced by $n-k+1$ and the same conclusions are derived. Recall that a Riemannian manifold M has *positive k-th Ricci curvature* if for each $p \in M$ and any $k+1$ orthonormal tangent vectors $e_0, e_1, \dots, e_k$ at p, the sum of sectional curvatures $\sum_{i=1}^{k} K(e_0, e_i) > 0$ [37]. The main examples with $k > 1$ are compact locally symmetric spaces with rank ≥ 2.

The following corollary of Theorem 1.6 is an immediate consequence of [35, Theorem 7].

Corollary 1.7. *The orbit space X is homeomorphic to the join of an $(f-1)$-simplex and the space (containing B) given by the intersection of all faces, where $f \leq \dim X$ is the number of faces of X.*

1.6. Outline of proof of Theorem 1.6

The basic idea is to construct a certain normal subgroup of G that contains all isotropy groups associated to codimension one strata of X and prove that its fixed point set is non-empty. Suppose first G is a simple Lie group. An involutive inner automorphism of G defines a symmetric space of inner type G/K and indeed corresponds to the geodesic symmetry at the base point of G/K. On one hand, we can estimate the codimension of the fixed point set of the involution in M, if we choose it to fix a regular point or an important point (i.e. a point projecting to a codimension one stratum of X), which we can always do. On the other hand, a finite number (which can be estimated in terms of the geometry of G/K) of conjugates of the involution generate a dense subgroup of G (this is because they correspond to geodesic symmetries of G/K at generic points, and these will generate sufficient transvections of G/K). Combining these two observations yields, via Frankel's Theorem, an estimate on the codimension of the fixed point set of G, which is thus non-empty if the dimension of M is sufficiently large. In the case of a general compact connected Lie group, the argument is more technical and one proceeds by induction using the simple factors and the center.

1.7. The Abelian case

We illustrate some ideas in the proof in the much simpler case of a torus action. So let a torus T^k act effectively and isometrically on an orientable connected complete n-manifold M of positive sectional curvature and assume $n \geq 2k$. Note that the principal isotropy group T_{pr} is trivial, since it is a normal subgroup. If p is an important point, T_p is an Abelian group that acts simply transitively on the unit sphere of the non-trivial component of the slice representation, and hence $T_p = S^0$ or $T_p = S^1$; the first case cannot occur, as the non-trivial element in $T_p = S^0$ would act as a reflection on a codimension one hypersurface of M and this is forbidden by the orientability of M. We choose a point for each codimension one stratum in X and end up with points $p_1, \ldots, p_\ell$. Let $L = T_{p_1} \cdots T_{p_\ell}$ be the group generated by the $T_{p_i} = S^1$. Since T is Abelian, the codimension of the fixed point set of T_{p_i} is 2. Owing to Frankel's Theorem, $\dim M^L \geq \dim M - 2\dim L \geq 2\dim T/L \geq 0$, so $M^L \neq \varnothing$. Let $\tilde{B}$ be a component of M^L of maximal dimension. Now T/L acts on $\tilde{B}$ and $\dim \tilde{B} \geq 2\dim T/L$. If $\partial(\tilde{B}/T) \neq \varnothing$, we can repeat the procedure; since $\dim T/L < \dim T$, the procedure must eventually stop.

We obtain a subtorus S of T containing L and hence all isotropy groups of codimension one strata of X, whose fixed point set M^S has a component B such that $\partial(B/T) = \varnothing$.

1.8. Example

Let $T^2 = S^1_1 \times S^1_2$ act on $M = S^5(1)$ by $(S^1_1, \mathbb{R}^2) \times (S^1_2, \mathbb{R}^2 \oplus \mathbb{R}^2)$, namely, (standard action) $\times$ (Hopf action). Then $X = S^3_+(\frac{1}{2})$, $\partial X = S^2(\frac{1}{2})$, $N = S^1_1$, $B = M^N = S^3(1)$ and $B/T^2 = \partial X$.

1.9. Structure of the paper

After a short section on preliminaries, we show in section 3 that the presence of boundary in the orbit space of the action implies the existence of certain *nice involutions*, whose codimension of the fixed point set we can estimate (Lemma 3.1), unless some special situation occurs. This is followed by section 4 in which a problem of independent interest about the geometry of symmetric spaces is investigated, namely, we want to know how many geodesic symmetries of a compact symmetric space are needed to generate a dense subgroup of the transvection group (compare Proposition 4.2 and Theorem 4.6). In sections 5 and 6, we apply the results of the two previous sections to prove Theorems 1.1 and 1.6, respectively. Section 7 is devoted to establishing conditions under which a reducible representation can have orbit space with non-empty boundary (Proposition 7.1 and Corollary 7.3). The proofs of our applications are finally collected in section 8.

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2. Preliminaries

Let G be a compact Lie group of isometries of a connected complete orientable Riemannian manifold M. Let X be the orbit space M/G equipped with the induced quotient metric. We generally assume that the action is effective.

The subset of M consisting of all points with isotropy groups conjugate to G_p is a submanifold of M, denoted by $M_{(G_p)}$, called an *isotropy stratum* of M, and projects to a Riemannian totally geodesic submanifold of X denoted $X_{(G_p)}$, called an *isotropy stratum* of X, which contains the point $x = Gp$.

Locally at $p \in M$, the orbit decomposition of M is completely determined by the *slice representation* of G_p on the normal space $\nu_p(Gp)$. The set of G_p-fixed vectors in $\nu_p(Gp)$ is tangent to $M_{(G_p)}$, and the action on its orthogonal complement in $\nu_p(Gp)$ has cohomogeneity equal to the codimension of $X_{(G_p)}$ in X.

A point $p \in M$ is called *regular* if the slice representation at p is trivial. It is called *exceptional* if it is not regular and the slice representation has discrete orbits. If it is neither regular nor exceptional, it is called *singular*. The set M_{reg} of all regular points in M is open and dense, and X_{reg} is connected and convex. X_{reg} is the stratum corresponding to the unique conjugacy class of minimal appearing isotropy groups; these are called *principal isotropy groups*.

The *boundary* of X is the closure of the union of all strata of codimension 1 in X. It is denoted by ∂X. A point $p \in M$ is projected to a stratum of codimension 1 in X if and only if the non-trivial component of the slice representation has cohomogeneity 1; we will call such points G*-important*.

We recall the easy but perhaps not much noticed fact that the components of the fixed point set of a connected group of isometries of an orientable manifold are *orientable* (closed totally geodesic) submanifolds [38, Theorem 3.5.2].

3. Nice involutions

Under the assumptions of section 2, a *nice involution* is a non-central element $\sigma \in G$ whose square σ^2 is in the center of G and whose fixed point in M is non-empty and has a component of codimension at most $c + \dim G/K$, with $c \leq 4$, where $K = G^\sigma$ is the centralizer of σ. Nice involutions will play an important role in estimating the codimensions of fixed point sets of certain groups of isometries of M.

Regarding the terminology in the statement of the next lemma, recall that, in the case of finite principal isotropy groups, along each component of a codimension one stratum of the orbit space, the connected slice representation is equivalent to one of $(\mathbb{Z}_2, \mathbb{R})$, $(S^1, \mathbb{C})$ or $(S^3, \mathbb{H})$, up to a trivial subrepresentation (see the proof of the lemma and compare [12, section 4]); we call the corresponding component respectively of $\mathbb{Z}_2$-, S^1- or S^3-type.

Lemma 3.1. *Assume G is a compact connected Lie group and $\partial X \neq \varnothing$. Then nice involutions exist unless the following situation $(\mathcal{S})$ is present: the principal isotropy group is finite of odd order, all boundary components of X are of S^1-type, and the identity components of their isotropy groups are contained in the center of G.*

Proof. Assume the situation $(\mathcal{S})$ does not happen. We will look for certain involutions $\sigma \in G_p$ such that $p \in M$ projects to a stratum of codimension at most 1 in X, and later estimate the codimension of their fixed point sets, proving that they are nice involutions. As we will see, in certain cases there are different kinds of possible choices for σ.

Fix a principal isotropy group H. Note that any element of H which is central in G belongs to all principal isotropy groups and thus lies in the kernel of the G-action, which we have assumed to be trivial.

Assume first that H is finite. For any G-important point $p \in M$, the isotropy group G_p acts transitively on the unit sphere S^a in the non-trivial component of the slice representation. It follows that G_p/H is diffeomorphic to S^a. In particular, for $a \geq 1$ there is a finite covering $G_p^0 \to S^a$. If $a \geq 2$, S^a is simply-connected, so the covering is a diffeomorphism and thus a equals 3. We can take $\sigma = -1 \in S^3 \approx G_p^0$, or a square root of this element if it is central in G. If $a = 1$, then G_p^0 is a finite covering of S^1, hence, $G_p^0 \approx S^1$ and we may assume this subgroup is non-central (thanks to our assumption that $(\mathcal{S})$ is not present). Then $Z(G) \cap G_p^0$ is at most a cyclic group and again we can take σ to be a square root of an element of $Z(G) \cap G_p^0$. If $p \in M$ is a G-important point with $G_p/H \approx S^0 = \mathbb{Z}_2$, there is $\sigma' \in G_p^0$ acting as -1 on the 1-dimensional non-trivial component of the slice representation. Since G is connected and M is orientable, σ' cannot be central. Since $(\sigma')^2$ is trivial on the slice, it is an element of H. In case H has odd order, also $(\sigma')^2$ has odd order, say $2b+1$, and we can take $\sigma = (\sigma')^{2b+1}$.

If H is finite with even order or $\dim H > 0$, it is clear that we can find an element $\sigma \in H$ of order 2. If possible, this is our preferred choice of σ, in terms of obtaining a better estimate on the codimension of M^σ, see below.

It remains only to estimate the codimension of the fixed point set of σ. The non-trivial component of the slice representation at p has dimension c equal to 0, 1, 2 or 4 according to whether p is a regular point or an important point projecting to a boundary stratum of type $\mathbb{Z}_2$, S^1 or S^3, respectively. Along the tangent space $T_p(Gp)$, the codimension of the fixed point set of σ is bounded by the dimension of the (-1)-eigenspace of Ad_σ, that is, $\dim G/K$. Hence the component through p of the fixed point set M^σ has codimension at most $c + \dim G/K$. □

Remark 3.2. The proof of the lemma shows that we can take $c = 0$ if $\dim H > 0$ or H is finite with even order, $c = 2$ if H is finite and there are no S^3-boundary components, and $c = 4$ in general.

4. Generic totally geodesics submanifolds of compact symmetric spaces

The second tool that we will use is an invariant attached to a compact connected symmetric space M. Define ℓ_M to be the minimal number ℓ such that there exists $p_1 \dots, p_\ell \in M$ "spanning" M, in the sense that these points do not lie in a proper connected closed totally geodesic submanifold of M.

More specifically, let $p_1, \dots, p_k \in M$ with $k \geq 2$ be generic points in the sense that each pair (p_i, p_j) with $i \neq j$ is connected by a unique shortest geodesic. In this case, it is clear that the intersection of all closed connected totally geodesic submanifolds of M which contain $p_1, \dots, p_k$ has a connected component containing $p_1, \dots, p_k$, which we call the *span* of $p_1, \dots, p_k$ and denote by $\langle p_1, \dots, p_k \rangle$. It is easy to see that ℓ_M is the minimal number ℓ such that there exist generic points $p_1, \dots, p_\ell \in M$ with $\langle p_1, \dots, p_\ell \rangle = M$.

Note also that $\langle p_1, \dots, p_k \rangle$ equals $\exp_{p_1}(T_{p_1}\langle p_1, \dots, p_k \rangle)$, where $T_{p_1}\langle p_1, \dots, p_k \rangle$ coincides with the intersection of all Lie triple systems in $T_{p_1}M$ containing $v_2, \dots, v_k$, where v_i is tangent to the geodesic joining p_1 to p_i. We deduce that $\ell_M = \ell_{M'}$ for a Riemannian covering $M' \to M$.

Consider now the case $k = 2$. It is clear that $\langle p_1, p_2\rangle$ is either a closed geodesic or the closure of a non-periodic infinite geodesic, that is, in any a case a flat torus of M. The extreme case occurs when p_2 is a regular point with respect to the isotropy action at p_1, and the geodesic through p_1 and p_2 is dense in the unique maximal flat torus T_{12} of M containing those points; in this case $\langle p_1, p_2\rangle = T_{12}$. In particular $\langle p_1, \dots, p_k\rangle$ for $k \geq 2$ has maximal rank and $\ell_M \geq 3$ if M is not flat. Indeed we shall see that $\ell_M = 3$ if M is an irreducible compact symmetric space of inner type, unless M is one of $\mathbb{H}P^2$, $\mathbb{O}P^2$ or $\mathsf{Gr}_k(\mathbb{K}^n)$ with $n > 3k$ (here $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$).

From now on, we impose further genericity conditions. Let $p_1, \dots, p_k \in M$ with $k \geq 2$ be generic points in the sense that each pair (p_i, p_j) with $i \neq j$ is connected by a unique shortest geodesic, p_j is regular with respect to the isotropy action at p_i so that p_i and p_j are contained in a unique maximal flat torus T_{ij} of M, and the product of the geodesic symmetries of M at p_i, p_j is a transvection generating a group acting transitively on a dense subset of T_{ij}. With these genericity conditions, denote by $L = L_{p_1,\dots,p_k}$ the closure of the group consisting of even products of geodesic symmetries of M at $p_1, \dots, p_k$. Then L is connected and $L(p_1) = \cdots = L(p_k)$ is a submanifold of $\langle p_1, \dots, p_k\rangle$. Since the geodesic symmetry of M at any point of $L(p_1)$ leaves this submanifold invariant, $L(p_1)$ is totally geodesic and hence $\langle p_1, \dots, p_k\rangle = L(p_1)$.

Write $M = G/K$ where G is the identity component of the isometry group of M (the transvection group of M) and $K = G_{p_1}$, and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the decomposition into the ± 1-eigenspaces of the involution induced by the geodesic symmetry at p_1; here $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{p} \cong T_{p_1}M$. Fix a maximal Abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. Then there are decompositions

$$\mathfrak{k} = \mathfrak{k}_0 + \sum_{\lambda\in\Lambda^+} \mathfrak{k}_\lambda, \quad \mathfrak{p} = \mathfrak{a} + \sum_{\lambda\in\Lambda^+} \mathfrak{p}_\lambda, \tag{4.3}$$

where Λ is a (possibly non-reduced) root system. The *marked Dynkin diagram of M* is the Dynkin diagram of Λ where each vertex is labeled by the multiplicity $m_\lambda = \dim \mathfrak{p}_\lambda$, with a special rule in case of non-reduced roots, see [24, p. 118].

Suppose p_1, $p_2 \in M$ are generic points and $\langle p_1, p_2\rangle = \exp_{p_1}(\mathfrak{a})$. A generic choice of $p_3 \in M$ corresponds under $\exp_{p_1}$ to $v_3 \in T_{p_1}M \cong \mathfrak{p}$ such that all $\mathfrak{p}_\lambda$-components of v_3 are nonzero, since the linear isotropy representation of K on $\mathfrak{p}$ preserves the $\mathfrak{p}_\lambda$. It follows that for generic $p_1, \dots, p_k \in M$ the marked Dynkin diagram of $\langle p_1, \dots, p_k\rangle$ as a symmetric space already coincides with that of M if $k = 3$, up to the multiplicities which are bounded above by those of M, and hence also for $3 < k < \ell_M$.

Below we compute ℓ_M for compact irreducible symmetric spaces of inner type. The reducible case is covered by the following lemma.

Lemma 4.1. *Let M_1 and M_2 be two compact symmetric spaces. Then*

$$\ell_{M_1\times M_2} = \max\{\ell_{M_1}, \ell_{M_2}\}.$$

Proof. Given generic points $p_1, \dots, p_k \in M_1 \times M_2$ with $k \geq 2$, the closed totally geodesic submanifold $\langle p_1, \dots p_k \rangle$ has maximal rank in $M_1 \times M_2$, so it is of the form $N_1 \times N_2$, where N_i is a totally geodesic submanifold of M_i. The result follows. □

The importance of the invariant ℓ_M for us lies in the following proposition.

Proposition 4.2. *Let G be a compact connected Lie group and assume $\sigma \in G$ is a non-central element whose square is central. Let $K = G^\sigma$ be the centralizer of σ. Assume G acts almost effectively on the symmetric space $M = G/K$. Then there exist $g_1, \dots, g_{\ell_M}$ such that the group generated by $g_1 \sigma g_1^{-1}, \dots, g_{\ell_M} \sigma g_{\ell_M}^{-1}$ is dense in G.*

Proof. Note that the assumption that G acts almost effectively on M says that σ does not centralize a normal subgroup of G of positive dimension and G is the identity component of the isometry group of M, up to a finite covering. Let p denote the base point of $M = G/K$ and choose $g_1, \dots, g_k$ such that $g_1 p, \dots, g_k p \in M$ are in generic position. Then the totally geodesic submanifold

$$N := \langle g_1 p, \dots, g_k p \rangle$$

is closed and connected, and the closure of the group generated by $g_1 \sigma g_1^{-1}, \dots, g_k \sigma g_k^{-1}$ is a closed subgroup of G containing all transvections of N; But $N = M$ for $k = \ell_M$. □

For the sake of computation of ℓ_M, we next introduce another invariant of a compact symmetric space $M = G/K$, where G is the transvection group of M. Define h_M to be the maximal number h such that the principal isotropy group H of the diagonal action of G on the h-fold product M^h is non-trivial. Note that for $h = 1$ the principal isotropy group is K, and for $h = 2$ the principal isotropy group is the principal isotropy group K_{pr} of the linear isotropy representation of K on the tangent space $T_p M$ at the base point p, which is never trivial, so $h_M \geq 2$. Indeed $h_M = 1 + \tilde{h}_M$, where $\tilde{h}_M \geq 1$ is the maximal number $\tilde{h}$ such that the principal isotropy group of the diagonal action of K on the $\tilde{h}$-fold sum $\oplus^{\tilde{h}} T_p M$ is non-trivial.

Proposition 4.3. *Let $M = G/K$ and H be as above. Then $h_M + 1 \leq \ell_M \leq h_M + 2$. Furthermore, in case $\ell_M = h_M + 2$ there is a closed connected totally geodesic submanifold N_2 of M (different from M) of codimension at most $\dim H$ such that $N_2 = \langle p_1, \dots, p_{\ell_M - 1} \rangle$ for generic points $p_1, \dots, p_{\ell_M - 1} \in M$. In particular, if H is finite then $\ell_M = h_M + 1$.*

Proof. Given generic points $p_1, \dots, p_h \in M$, $h = h_M$, they all lie in M^H, up to replacing H by a conjugate group. Note that M^H is a closed totally geodesic submanifold of M, but it is not necessarily connected. However, H centralizes the geodesic symmetries at the p_i and hence centralizes the group $L_{p_1, \dots, p_h}$. It follows that $\langle p_1, \dots, p_h \rangle \subset M^H$. Since the former submanifold is connected, we deduce that $\ell_M - 1 \geq h_M$.

It remains to obtain the upper bound for ℓ_M. Assume $\ell = \ell_M > h+1$ and fix generic points $p_1, \dots, p_{\ell-2}, q_{\ell-1} \in M$. We have the closed connected totally geodesic submanifolds $N_1 = \langle p_1, \dots, p_{\ell-2} \rangle$ and $N_2 = \langle p_1, \dots, p_{\ell-2}, q_{\ell-1} \rangle$. Owing to the fact that the number of closed connected totally geodesic submanifolds of a compact symmetric space, up to congruence, is countable, there is a subset of positive measure U of M such that for all $p_{\ell-1} \in U$, the flag of closed connected totally geodesic submanifolds $N_1 \subset \langle p_1, \dots, p_{\ell-1} \rangle$ is G-conjugate to $N_1 \subset N_2$. In other words, for all $p_{\ell-1} \in U$ there is $\iota \in G$ such that $\iota(p_i) \in N_1$ for $i \leq \ell - 2$ and $\iota(p_{\ell-1}) \in N_2$.

The isometry group $\mathrm{Iso}(N_1)$ is a compact Lie group with finitely many connected components. By [34, Lemma 7.5], there is a finite subgroup F of $\mathrm{Iso}(N_1)$ meeting every component. We can find ψ in the identity component $\mathrm{Iso}_0(N_1)$ such that $\psi \cdot \iota|_{N_1} \in F$. Every geodesic symmetry of N_1 uniquely extends to a geodesic symmetry of N_2 and then to a geodesic symmetry of M, and hence every transvection of N_1 admits an extension (not necessarily unique) to a transvection of N_2 and then to a transvection of M. Since the group generated by transvections at a fixed point of a symmetric space coincides with the identity component of the isometry group of the symmetric space, we may consider $\psi \in G$ and then the element $\psi \cdot \iota \in G$ also maps $p_{\ell-1}$ to N_2. We have shown that we can always take $\iota \in \hat{H}$, where

$$\hat{H} := \{ g \in G \mid g(N_1) = N_1 \text{ and } g|_{N_1} \in F \}.$$

Notice that $\hat{H}$ is closed subgroup of G and hence a Lie group, with the same identity component as the isotropy group $\bar{H}$ of $(p_1, \dots, p_{\ell-2})$. Since U has positive measure and $\hat{H}(N_2) \supset U$, it follows that the codimension of N_2 is bounded above by $\dim \hat{H} = \dim \bar{H}$. In particular $\bar{H}$ is nontrivial and hence $h = \ell - 2$ and $\bar{H} = H$. □

Corollary 4.4. *If every maximal connected closed totally geodesic submanifold of M is given as a component of the fixed point set of a subgroup of G, then $\ell_M = h_M + 1$.*

Proof. Given generic points $p_1, \dots, p_{\ell_M - 1} \in M$, there exists a connected closed totally geodesic submanifold N containing those points, and we can assume N is maximal. By assumption, N is a component of the fixed point set of a non-trivial subgroup H of G. Now H is contained in the isotropy group $\tilde{H}$ of a generic $(\ell_M - 1)$-tuple of points of M, so $h_M \geq \ell_M - 1$. □

Remark 4.5. A connected totally geodesic submanifold of M is called *reflective* if it is a connected component of the fixed point set of an involutive isometry of M; if, in addition, the involutive isometry can be taken in the transvection group of M, then the submanifold will be called *inner reflective*. It follows from Corollary 4.4 that if every maximal connected closed totally geodesic submanifold of M is inner reflective, then $\ell_M = h_M + 1$.

Table 3
The invariant ℓ_M for some irreducible symmetric spaces of compact type.

$M = G/K$	ℓ_M	Conditions
$\mathsf{SO}(n)/(\mathsf{SO}(p)\times\mathsf{SO}(n-p))$	$\max\{3, \lceil \frac{n}{p} \rceil\}$	$p \leq n/2$
$\mathsf{SU}(n)/\mathsf{S}(\mathsf{U}(p)\times\mathsf{U}(n-p))$		$p \leq n/2$
$\mathsf{Sp}(n)/(\mathsf{Sp}(p)\times\mathsf{Sp}(n-p))$		$p \leq n/2,\ (n,p) \neq (3,1)$
$\mathsf{Sp}(3)/\mathsf{Sp}(1)\times\mathsf{Sp}(2)$	4	–
$\mathsf{F}_4/\mathsf{Spin}(9)$		
$\mathsf{Sp}(n)/\mathsf{U}(n)$	3	–
$\mathsf{SO}(2n)/\mathsf{U}(n)$		$n \geq 5$
$\mathsf{G}_2/\mathsf{SO}(4)$		–
$\mathsf{F}_4/\mathsf{Sp}(3)\mathsf{Sp}(1)$		–
$\mathsf{E}_6/\mathsf{Spin}(10)\mathsf{U}(1)$		–
$\mathsf{E}_6/\mathsf{SU}(6)\mathsf{SU}(2)$		–
$\mathsf{E}_7/\mathsf{E}_6\mathsf{U}(1)$		–
$\mathsf{E}_7/(\mathsf{SU}(8)/\mathbb{Z}_2)$		–
$\mathsf{E}_7/\mathsf{Spin}(12)\mathsf{SU}(2)$		–
$\mathsf{E}_8/\mathsf{Spin}(16)$		–
$\mathsf{E}_8/\mathsf{E}_7\mathsf{SU}(2)$		–

Theorem 4.6. *The invariant ℓ_M for various irreducible symmetric spaces M of compact type is listed in Table 3, including all spaces of inner type.*

Proof. We run through the cases.

Symmetric spaces of maximal rank. They are

$$\mathsf{SO}(2p)/(\mathsf{SO}(p)\times\mathsf{SO}(p)),\ \mathsf{SU}(n)/\mathsf{SO}(n),\ \mathsf{Sp}(n)/\mathsf{U}(n),$$
$$\mathsf{SO}(2p+1)/(\mathsf{SO}(p+1)\times\mathsf{SO}(p)),\ \mathsf{E}_6/(\mathsf{Sp}(4)/\mathbb{Z}_2),\ \mathsf{E}_7/(\mathsf{SU}(8)/\mathbb{Z}_2),$$
$$\mathsf{E}_8/\mathsf{SO}'(16),\ \mathsf{F}_4/\mathsf{Sp}(3)\mathsf{Sp}(1) \text{ and } \mathsf{G}_2/\mathsf{SO}(4)$$

(not all listed in Table 3). The condition $\operatorname{rk} M = \operatorname{rk} G$ is equivalent to the effective K_{pr} being finite [24, Proposition 4.1] (and indeed isomorphic to $\mathbb{Z}_2^{\operatorname{rk} M}$). Therefore $h_M = 2$ and $\ell_M = 3$ by Proposition 4.3.

The Cayley projective plane $\mathbb{O}P^2 = \mathsf{F}_4/\mathsf{Spin}(9)$. The linear isotropy representation of $K = \mathsf{Spin}(9)$ on $\mathbb{R}^{16}$ has $K_{pr} = \mathsf{Spin}(7)$ and corresponding K_{pr}-irreducible decomposition $\mathbb{R}\oplus\mathbb{R}^7\oplus\mathbb{R}^8$. The principal isotropy group of this action is $H \cong \mathsf{SU}(3)$, with corresponding decomposition $4\mathbb{R}\oplus 2\mathbb{C}^3$. The principal isotropy group of this action is trivial, so $h_M = 3$. A maximal closed connected totally geodesic submanifold of $\mathbb{O}P^2 = \mathsf{F}_4/\mathsf{Spin}(9)$ is either $\mathbb{O}P^1 = \mathsf{Spin}(9)/\mathsf{Spin}(8)$ or $\mathbb{H}P^2 = (\mathsf{Sp}(3)\cdot\mathsf{Sp}(1))/(\mathsf{Sp}(2)\cdot\mathsf{Sp}(1)\cdot\mathsf{Sp}(1))$. Since $\mathsf{Spin}(9)$ and $\mathsf{Sp}(3)\cdot\mathsf{Sp}(1)$ are components of fixed point sets of inner automorphisms of F_4, $\mathbb{O}P^1$ and $\mathbb{H}P^2$ are inner reflective. We deduce from Corollary 4.4 that $\ell_M = 4$.

Grassmann manifolds. Let $M = \mathsf{Gr}_r(\mathbb{K}^n)$ with $n \geq 2r$, where $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Given $p_1,\dots,p_k \in M$, these points respectively lift to r-dimensional $\mathbb{K}$-subspaces $\pi_1,\dots,\pi_k \subset \mathbb{R}^n$. If $k < \frac{n}{r}$, then clearly the span of $\pi_1,\dots,\pi_k$ is a proper subspace of $\mathbb{K}^n$

and $p_1, \dots, p_k \in \mathrm{Gr}_q(\mathbb{K}^{kr})$, so that $\ell_M - 1 \geq k$. Note that we can always take $k = m-1$, where $m := \lceil \frac{n}{r} \rceil \geq 2$, so $\ell_M \geq \max\{3, m\}$.

If $m = 2$, then $n = 2r$ (in the case $\mathbb{K} = \mathbb{R}$, M is a space of maximal rank and this case has already been examined) and it is not so difficult to find three points in M not contained in a proper connected closed totally geodesic submanifold, implying $\ell_M = 3$. Next we assume $m \geq 3$ and want to show that there exist $p_1, \dots, p_m \in M$ such that $N := \langle p_1, \dots, p_m \rangle$ coincides with M. This will prove $\ell_M = m$. Let $\{e_i\}_{i=1}^n$ be the canonical $\mathbb{K}$-basis of $\mathbb{K}^n$ and consider $p_1, \dots, p_m$ associated to the r-dimensional subspaces (note that $(m-1)r + 1 \geq n - r + 1$):

$$
\begin{aligned}
\pi_1 &= \mathrm{span}(e_1, \dots, e_r), \\
&\vdots \\
\pi_{m-1} &= \mathrm{span}(e_{(m-2)r+1}, \dots, e_{(m-1)r}), \\
\pi_m &= \mathrm{span}(e_{n-r+1}, \dots, e_n).
\end{aligned}
$$

By slightly perturbing the points $p_1, \dots, p_m$, we can ensure that N is a connected closed totally geodesic submanifold of maximal rank and same restricted root system as M. In cases $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, using the classification [24, p. 119 and p. 146] this already implies that N is a $\mathbb{K}$-Grassmannian. In case $\mathbb{K} = \mathbb{H}$, below we distinguish between $r > 1$ and $r = 1$ to prove that N is an $\mathbb{H}$-Grassmannian. In any case, since $\{e_i\}_{i=1}^n$ has been perturbed to another $\mathbb{K}$-basis of $\mathbb{K}^n$, we must have $N = M$.

In case $\mathbb{K} = \mathbb{H}$ and $r > 1$ we check that, for generic $p_1, \dots, p_m$, N is an $\mathbb{H}$-Grassmannian as follows. Consider the restricted root space decomposition (4.3) where p_1 is the basepoint and $\langle p_1, p_2 \rangle = \exp_{p_1}(\mathfrak{a})$. It is not difficult to see that $v_3 \in T_{p_1}M \cong \mathfrak{p}$ can be chosen so that $\langle p_1, p_2, p_3 \rangle$ is not a $\mathbb{C}$-Grassmannian. Note also that $N = \mathsf{SO}(4r+2)/\mathsf{U}(2r+1)$ is not a totally geodesic submanifold of M (one way to see that is as follows: consider the restricted root system $\{\theta_i \pm \theta_j, \theta_i, 2\theta_i\}$ of type BC_r; of course $[\mathfrak{p}_{\theta_1+\theta_2}, \mathfrak{p}_{\theta_1-\theta_2}] \subset \mathfrak{k}_{2\theta_1} + \mathfrak{k}_{2\theta_2}$; one computes directly that the left hand-side has dimension 3 in case of M, and hence also in case of N as the multiplicities of $\theta_1 \pm \theta_2$ equal 4 in both cases; however the right hand-side has dimension 2 in case of N). It follows from the classification [24, p. 119 and p. 146] that $\langle p_1, p_2, p_3 \rangle$ is an $\mathbb{H}$-Grassmannian and so is $\langle p_1, \dots, p_m \rangle$.

In case $\mathbb{K} = \mathbb{H}$ and $r = 1$, $M = \mathbb{H}P^{n-1}$ is of type BC_1 and $m = n$. It is not difficult to see that for a generic choice of points, $\langle p_1, p_2, p_3 \rangle = \mathbb{C}P^2$. If $n = 3$, this is a maximal totally geodesic submanifold, so $\ell_{\mathbb{H}P^2} = 4$. If $n > 3$, $\langle p_1, p_2, p_3, p_4 \rangle$ is an $\mathbb{H}$-projective space, and so is $\langle p_1, \dots, p_m \rangle$. We finish as above to deduce that $\ell_M = m = n$.

The space $\mathsf{SO}(4n)/\mathsf{U}(2n)$. Note that the cases $n = 1$ and $n = 2$ are respectively locally isometric to a sphere and a real Grassmannian, so we may assume $n \geq 3$. The linear isotropy representation is $\Lambda^2\mathbb{C}^{2n}$ with $K_{pr} = \mathsf{SU}(2)^n$, so $h_M = 2$. If $\ell_M = 4$ then due to Proposition 4.3 M contains a connected closed totally geodesic submanifold N_2 with the same Dynkin diagram, and dimension at least $(4n^2 - 2n) - 3n = 4n^2 - 5n$. According to [24, p. 119 and 146], the submanifold with same diagram, not larger multiplicities and maximal dimension is $\mathsf{Gr}_n(\mathbb{C}^{2n})$, which has dimension $2n^2 < 4n^2 - 5n$ for $n \geq 3$. Hence $\ell_M = 3$.

The space $\mathsf{SO}(4n+2)/\mathsf{U}(2n+1)$. Note that the case $n = 1$ is locally isometric to a $\mathbb{C}P^3$, so we may assume $n \geq 2$. The linear isotropy representation is $\Lambda^2\mathbb{C}^{2n+1}$ with $K_{pr} = \mathsf{SU}(2)^n\mathsf{U}(1)$, so $h_M = 2$. If $\ell_M = 4$ then due to Proposition 4.3 M contains a connected closed totally geodesic submanifold N_2 the same Dynkin diagram, and dimension at least $(4n^2+2n)-(3n+1) = 4n^2-n-1$. Note that $\mathsf{SU}(2n+2)$ is not a subgroup of $\mathsf{SO}(4n+2)$ so, according to [24, p. 119 and 146], the only candidate with same diagram, not larger multiplicities and maximal dimension is $\mathsf{Gr}_n(\mathbb{C}^{2n+1})$ which, however, has dimension $2n^2+2n < 4n^2 - n - 1$ for $n \geq 2$. Hence $\ell_M = 3$.

The space $\mathsf{E}_6/\mathsf{Spin}(10)\mathsf{U}(1)$. The linear isotropy representation is $\mathbb{C}^{16} \otimes_{\mathbb{C}} \mathbb{C}$ with $H = K_{pr} = \mathsf{U}(4)$ and corresponding decomposition $4\mathbb{R}\oplus 2\mathbb{C}^4\oplus 2\mathbb{R}^6$, so $h_M = 2$. If $\ell_M = 4$ then, due to Proposition 4.3, M contains a connected closed totally geodesic submanifold $N_2 = \langle p_1, p_2, p_3\rangle$ of rank 2, same Dynkin diagram, and dimension at least $\dim M - \dim H = 32 - 16 = 16$; here p_1, p_2, $p_3 \in M$ are generic points. According to Chen and Nagano (cf. [6, 22]), the submanifolds under these conditions are $\mathsf{Gr}_2(\mathbb{C}^6)$ and $\mathsf{SO}(10)/\mathsf{U}(5)$. The first submanifold is a connected component of the fixed point set of the geodesic symmetry of $\mathsf{E}_6/\mathsf{SU}(6)\mathsf{SU}(2)$ (compare [22, p. 1115] and [23, Proposition 3.5]). Similarly, the second one is a polar submanifold, namely, a connected component of the fixed point set of the geodesic symmetry of M, see [22, p. 1119]. It follows that in both cases the isotropy group in E_6 of a generic triple of points in M is non-trivial, which is a contradiction to $h_M = 2$ (compare Corollary 4.4). Hence $\ell_M = 3$.

The space $\mathsf{E}_6/\mathsf{SU}(6)\mathsf{SU}(2)$. The linear isotropy representation is $\Lambda^3\mathbb{C}^6 \otimes_{\mathbb{H}} \mathbb{C}^2$ with $K_{pr} = T^2 \cdot \mathbb{Z}_2$ [20, p. 436], so $h_M = 2$. If $\ell_M = 4$, then due to Proposition 4.3 M contains a connected closed totally geodesic submanifold N_2 of codimension 2, but this symmetric space is of index bigger than 2 [5, Thm. 1.2]. Hence $\ell_M = 3$.

The space $\mathsf{E}_7/\mathsf{E}_6\mathsf{U}(1)$. The linear isotropy representation is $\mathbb{C}^{27} \otimes_{\mathbb{C}} \mathbb{C}$ with $K_{pr} = \mathsf{Spin}(8)$, and $h_M = 2$. If $\ell_M = 4$ then M contains a connected closed totally geodesic submanifold N_2 of C_3-type, multiplicities bounded above by $(8, 8, 1)$ and dimension at least 26. Looking at the list of diagrams [24, p. 119], N_2 must be $\mathsf{SO}(12)/\mathsf{U}(6)$. This submanifold is a connected component of the fixed point set of the involution of $M = \mathsf{E}_7/\mathsf{E}_6\mathsf{U}(1)$ induced by the involution of E_7 defining $\mathsf{Spin}(12)\mathsf{SU}(2)$ as a symmetric subgroup (cf. [29, p. 70] and [23, Proposition 3.5]); since the latter is an inner automorphism of E_7, it follows as in the case of $\mathsf{E}_6/\mathsf{Spin}(10)\mathsf{U}(1)$ that the isotropy group of a generic triple of points of M is non-trivial, a contradiction to $h_M = 2$. Hence $\ell_M = 3$.

The space $\mathsf{E}_7/\mathsf{Spin}(12)\mathsf{SU}(2)$. The linear isotropy representation is $\mathbb{C}^{32} \otimes_{\mathbb{H}} \mathbb{C}^2$ with $K_{pr} = \mathbb{Z}_2^2 \cdot \mathsf{Sp}(1)^3$ [20, p. 436], so $h_M = 2$. If $\ell_M = 4$ then M contains a connected closed totally geodesic submanifold N_2 of rank 4, F_4-type, multiplicities bounded above by $(4, 4, 1, 1)$ and dimension at least 55. However, there exist no symmetric spaces under these conditions [24, p. 119 and 146].

The space $\mathsf{E}_8/\mathsf{E}_7\mathsf{SU}(2)$. The linear isotropy representation is $\mathbb{C}^{56}\otimes_{\mathbb{H}}\mathbb{C}^2$ with $K_{pr} = \mathbb{Z}_2^2\cdot\mathsf{Spin}(8)$ [20, p. 436], so $h_M = 2$. If $\ell_M = 4$ then M contains a connected closed totally geodesic submanifold N_2 of rank 4, F_4-type and dimension at least 84. Looking at the list of diagrams in [24], we see there are no submanifolds under these conditions. Hence $\ell_M = 3$. □

Remark 4.7. For each symmetric space G/K in Table 3, a direct check shows that the following estimates hold:

$$\ell_{G/K}\dim G/K \leq 2\dim G;$$
$$\ell_{G/K} \leq 2\,\mathrm{rk}\,G + 1.$$

These will be used in the proof of Theorem 1.6.

5. Proof of Theorem 1.1

Although the proof of this theorem is contained in the proof of Theorem 1.6, to be proved in the next section, it is instructive to do this proof first.

Let $\mathcal{L}_G$ be given by (1.1). Since G is assumed to simple, situation $(\mathcal{S})$ in Lemma 3.1 does not occur, so the lemma yields a nice involution $\sigma \in G$. Let $K = G^\sigma$. Then G acts almost effectively on the symmetric space G/K. By Proposition 4.2 we can find $g_1, \ldots, g_{\ell_{G/K}} \in G$, such that the group generated by $\sigma_i = g_i\sigma g_i^{-1}$ for $i = 1, \ldots, \ell_{G/K}$ is dense in G. Using Frankel's theorem and the codimension estimate for nice involutions, we obtain that

$$\begin{aligned}\dim M - \dim M^G &= \dim M - \dim M^{\sigma_1}\cap\cdots M^{\sigma_{\ell_{G/K}}}\\ &\leq \sum_{i=1}^{\ell_{G/K}} \dim M - \dim M^{\sigma_i}\\ &\leq \ell_{G/K}(4+\dim G/K)\\ &= \mathcal{L}_G,\end{aligned}$$

as desired. This completes the proof of the theorem.

6. Proof of the main result

We now proceed with the much more involved proof of Theorem 1.6. We follow a finite algorithm. At each step, there are two possibilities, namely, the situation $(\mathcal{S})$ as in Lemma 3.1 is present or not.

6.1. $(\mathcal{S})$ is not present

By Lemma 3.1, we can choose a nice involution $\sigma \in G$. Then G/K is a symmetric space of inner type, where $K = G^\sigma$, which locally splits as $G_1/K_1 \times \cdots \times G_m/K_m$, where each factor G_i/K_i is not necessarily irreducible, but instead $\ell_i := \ell_{G_i/K_i}$ satisfies $\ell_i < \ell_{i+1}$ for $i = 1, \ldots m-1$. Furthermore we may take G_i to be a connected closed normal subgroup of G acting with finite kernel on G_i/K_i.

Put $G' = G_1 \cdots G_m \subset G_{ss}$, non-trivial connected semisimple Lie group. Then $G = G' \cdot G''$ where G'' denotes the identity component of the centralizer of G' in G and contains $Z(G)^0$. It follows that $\alpha_G = \alpha_{G'} + \alpha_{G''}$ and $\beta_G = \beta_{G''}$.

We will construct a component $\tilde{B}$ of $M^{G'}$ such that $\dim \tilde{B} > \alpha_{G''} + \beta_{G''}$. The point here is, since $\tilde{B}$ is orientable and positively curved, in case $\tilde{B}/G''$ has non-empty boundary, we can check whether $(\mathcal{S})$ is present or not and repeat the argument for the action of G'' on $\tilde{B}$. In repeating the argument for G'', we get a similar decomposition $G'' = (G'')' \cdot (G'')''$ and construct a component $\tilde{\tilde{B}}$ of $(\tilde{B})^{(G'')'}$ on which $(G'')''$ acts etc. Note that both G' and $(G'')'$ act trivially on $\tilde{\tilde{B}}$. Since $\dim G'' < \dim G$, this process will stop after finitely many steps.

So each time that we repeat the argument, we get a typical pair $(G', \tilde{B})$. We construct an ascending chain of connected normal subgroups of G by collecting the factors G' in each step. The maximal element in this chain is a connected normal subgroup $G^\infty = G' \cdot (G'')' \cdot ((G'')'')' \cdots$ of G that will contain all isotropy groups associated to codimension 1 strata of X. On the other hand, as a fixed point set, in each step the $\tilde{B}$ form a descending chain of connected totally geodesic submanifolds of M, whose minimal element is a component B of M^{G^∞}, which will be contained in all faces of X. Finally, we enlarge G^∞ to the (possibly disconnected) subgroup N of G that fixes B pointwise.

Choose ℓ_m elements $g_1, \ldots, g_{\ell_m} \in G'$ in general position. Since for each i, $\ell_{(\cdot)}$ is the same number ℓ_i for all irreducible factors of G_i/K_i, we deduce from Remark 4.7 that $\ell_i \dim G_i/K_i \leq 2 \dim G_i$ for all i. It follows that

$$\begin{aligned} \ell_1 \dim G/K &= \sum_{i=1}^{m} \ell_i \dim G_i/K_i - \sum_{i=2}^{m} (\ell_i - \ell_1) \dim G_i/K_i \\ &\leq 2 \dim G' - \sum_{i=2}^{m} (\ell_i - \ell_1) \dim G_i/K_i. \end{aligned} \tag{6.4}$$

For each i, the fixed point set of the nice involution $\sigma_i := g_i \sigma g_i^{-1}$ has a component of codimension at most $4 + \dim G/K$. Denote by F_1 a component of maximal dimension of $M^{\sigma_1} \cap \cdots \cap M^{\sigma_{\ell_1}}$. As in section 5, F_1 is non-empty and

$$\dim F_1 \geq \dim M - \ell_1 (4 + \dim G/K). \tag{6.5}$$

Further, from Remark 4.7 we have $\ell_m \leq 2 \operatorname{rk} G_m + 1 \leq 2 \operatorname{rk} G' + 1$. We combine this inequality with estimates (6.4) and (6.5), and the assumption on $\dim M$, to write

$$\begin{aligned}\dim F_1 &> \alpha_G + \beta_G - \ell_1(4 + \dim G/K) \\ &\geq \alpha_{G''} + \beta_{G''} + 2\dim G' + 8\,\mathrm{rk}\, G' + 4 - 4\ell_1 - \ell_1 \dim G/K \\ &\geq \alpha_{G''} + \beta_{G''} + 4(\ell_m - \ell_1) + \sum_{i=2}^{m}(\ell_i - \ell_1)\dim G_i/K_i \\ &\geq 0.\end{aligned} \tag{6.6}$$

Note that σ does not centralize G_1. The closure of the group generated by σ_i for $i = 1, \dots, \ell_1$ contains the transvection group of a totally geodesic submanifold of $G_1/K_1 \times \cdots \times G_m/K_m$ of maximal rank, so it is locally a product and contains G_1. Therefore $F_1 \subset M^{G_1}$. Let B_1 be the component of M^{G_1} that contains F_1. Since G_1 is normalized by G and G is connected, G acts on B_1.

We next claim that for all $i \geq \ell_1 + 1$ the totally geodesic submanifolds M^{σ_i} and B_1 intersect along a submanifold of dimension at least $\dim B_1 - (4 + \dim G/K - \dim G_1/K_1)$. Note first that $F_1 \subset M^{\sigma_1} \cap B_1$, B_1 is G-invariant and $M^{\sigma_i} = g_i g_1^{-1} \cdot M^{\sigma_1}$, so $M^{\sigma_i} \cap B_1 \neq \varnothing$. In order to estimate the codimension of the intersection, consider the normal space of M^{σ_i} at a generic point q. Since G_1 is a normal subgroup of G, $\nu_q M^{\sigma_i}$ splits as a sum $V_q \oplus W_q$ where V_q is the part contained in $T_q(G_1 q)$ and W_q is its orthogonal complement. Going back to the argument in the last paragraph of the proof of Lemma 3.1, note that along $T_q(G_1 q)$ the codimension of M^{σ_i} is bounded by the dimension of the (-1)-eigenspace of Ad_{σ_i}, that is, $\dim G_1/K_1$, so $\dim V_q \leq \dim G_1/K_1$ and similarly $\dim W_q \leq 4 + \dim G/K - \dim G_1/K_1$. As a point $p \in M^{\sigma_i} \cap B_1$ is approached by generic points $q_n \in M^{\sigma_i}$, the numbers $\dim G_1 q_n$, $\dim V_{q_n}$ and $\dim W_{q_n}$ stay constant, say $\dim V_{q_n} = r$ and $\dim W_{q_n} = s$, and (passing to a subsequence) V_{q_n} converges to an r-dimensional subspace V_p of $\nu_p M^{\sigma_i}$. Since $T_p B_1 = (T_p M)^{G_1}$, we obtain that V_p is contained in $\nu_p B_1$. Now $\dim(\nu_p M^{\sigma_i} \cap \nu_p B_1) \geq r$. It follows that

$$\begin{aligned}\dim B_1 - \dim(M^{\sigma_i} \cap B_1) &\leq \dim T_p B_1 - \dim(T_p B_1 \cap T_p M^{\sigma_i}) \\ &= \dim(T_p B_1 + T_p M^{\sigma_i}) - \dim T_p M^{\sigma_i} \\ &= \dim \nu_p M^{\sigma_i} - \dim(\nu_p B_1 \cap \nu_p M^{\sigma_i}) \\ &\leq (r + s) - r \\ &\leq 4 + \dim G/K - \dim G_1/K_1.\end{aligned} \tag{6.7}$$

Let F_2 be a component of maximal dimension of $F_1 \cap M^{\sigma_{\ell+1}} \cap \cdots \cap M^{\sigma_{\ell_2}}$. By Frankel's theorem applied to B_1 as ambient space, $F_2 \neq \varnothing$. In fact, using (6.6) and (6.7), we obtain that

$$\begin{aligned}\dim F_2 &\geq \dim F_1 + \dim B_1 \cap M^{\sigma_{\ell_1+1}} \cap \cdots \cap M^{\sigma_{\ell_2}} - \dim B_1 \\ &> \alpha_{G''} + \beta_{G''} + 4(\ell_m - \ell_1) + \sum_{i=2}^{m}(\ell_i - \ell_1)\dim G_i/K_i\end{aligned}$$

$$
\begin{aligned}
&-(\ell_2-\ell_1)(4+\dim G/K-\dim G_1/K_1)\\
&=\alpha_{G''}+\beta_{G''}+4(\ell_m-\ell_1)+\sum_{i=2}^{m}(\ell_i-\ell_1)\dim G_i/K_i\\
&\quad -4(\ell_2-\ell_1)-(\ell_2-\ell_1)\sum_{i=2}^{m}\dim G_i/K_i\\
&=\alpha_{G''}+\beta_{G''}+4(\ell_m-\ell_2)+\sum_{i=3}^{m}(\ell_i-\ell_2)\dim G_i/K_i\\
&\geq 0.
\end{aligned}
$$

The closure of the subgroup generated by σ_i for $i = 1, \ldots, \ell_2$ contains G_1G_2. Therefore $F_2 \subset M^{G_1G_2}$. Let B_2 be the component of $M^{G_1G_2}$ that contains F_2. Note that G acts on B_2. Proceeding by induction, we find a component $\tilde{B} = B_m$ of $M^{G'}$ that contains a component F_m of maximal dimension of $M^{\sigma_1} \cap \cdots \cap M^{\sigma_{\ell_M}} \neq \varnothing$ of dimension

$$\dim \tilde{B} > \alpha_{G''} + \beta_{G''}.$$

Note that $\tilde{B}$ is orientable and the action of G'' on $\tilde{B}$ satisfies the dimension hypothesis in the statement of the theorem, so if $\tilde{B}/G''$ has non-empty boundary, we can check whether $(\mathcal{S})$ is present or not and continue the process.

6.2. $(\mathcal{S})$ is present

Then G_{pr} is finite and there is a G-important point $p \in M$ such that G_p^0 is a central circle group. Set $G' := G_p^0$. The fixed point set $M^{G'}$ has codimension 2 in M. Let $\tilde{B}$ be the component of $M^{G'}$ containing p. Then $\tilde{B}$ is orientable, $G'' := G/G'$ acts on $\tilde{B}$ and $\dim \tilde{B} = \dim M - 2 > \alpha_G + \beta_G - 2 = \alpha_{G''} + \beta_{G''}$, so if $\tilde{B}/G''$ has non-empty boundary, we can check whether $(\mathcal{S})$ is present or not and continue the process.

6.3. End of proof

In any case, G'' is a connected Lie group with $\dim G'' < \dim G$, so the process must stop after finitely many repetitions of the argument. We end up with a component B of the fixed point set of a normal subgroup G^∞ of G such that B/G has empty boundary. Let N be the subgroup of G consisting of all elements that fix B pointwise. It is clear that N is a (possibly disconnected) normal subgroup of G of positive dimension containing G^∞, $\dim B > \alpha_{G/N^0} + \beta_{G/N^0}$, the action of G/N on B is effective and its orbit space has empty boundary. In particular, the principal isotropy group of G/N on B is trivial by [35, Lemma 3.1]. This proves part (a) and the first statement of part (b).

Since $\dim B/G > 0$, the Frankel-Petrunin theorem for positively curved Alexandrov spaces [31, Theorem 3.2] implies that B/G meets each face of M/G. Since B/G itself has no codimension one strata, it follows that B/G is contained in each face of M/G. It follows that any isotropy group corresponding to a codimension one stratum of M/G is contained in the principal isotropy of the action of G on B, namely, N. This proves the second statement of part (b) and part (c)(i).

Since B/G is contained in the boundary of M/G, the isotropy (slice) representation of N at a generic point $p \in B$ has orbit space with non-empty boundary, which is part (c)(ii). Assume now M is simply-connected and let us show that the same holds for the isotropy representation of N^0 at p. There is a principal isotropy group G_{pr} contained in N. If $\dim G_{pr} > 0$, then $(G_{pr})^0 \subset N^0$. This implies that the isotropy representation of N^0 has non-trivial principal isotropy group and the desired result follows from [35, Lemma 3.1]. It remains to discuss the case in which G_{pr} is finite. Recall N contains all isotropy groups corresponding to codimension one strata of M/G. Owing to the simple-connectedness of M and [12, Lemma 3.6], there are no boundary components of $\mathbb{Z}_2$-type. Now N contains isotropy groups of dimensions 1 or 3 associated to codimension one strata of X, and then N^0 contain the corresponding identity components; these groups give rise to codimension one strata for the isotropy representation of N^0. This proves (c)(iii) and completes the proof of Theorem 1.6.

7. Reducible representations

The following proposition follows from [32, Proposition 12.1], but we provide a proof for the sake of clarity.

Proposition 7.1. *Let $\rho : G \to \mathsf{O}(V)$ be a representation of a compact Lie group G with orbit space $X = V/G$. Assume $V = V_1 \oplus V_2$ is a G-invariant decomposition, write $\rho = \rho_1 \oplus \rho_2$, denote a principal isotropy group of ρ_i by H_i, for $i = 1, 2$, and put $Y_1 = V_1/\rho_1(H_2)$ and $Y_2 = V_2/\rho_2(H_1)$. Then $\partial X \neq \varnothing$ if and only if H_2 is non-trivial and $\partial Y_1 \neq \varnothing$ or H_1 is non-trivial and $\partial Y_2 \neq \varnothing$.*

Proof. Let $p_1 \in V_1$ be a point with $G_{p_1} = H_1$. The slice representation of H_1 on $\nu_{p_1}(Gp_1)$ is the sum of a trivial component and $\rho_2|_{H_1}$. If $\partial Y_2 \neq \varnothing$, then the orbit space of the slice representation has non-empty boundary and hence p_1 projects to a point in ∂X.

Conversely, suppose $p = p_1 + p_2 \in V$ is a G-important point, where $p_i \in V_i$. Then the slice representation $(G_p, \nu_p := \nu_p(Gp))$ decomposes as the sum of a trivial component and a cohomogeneity 1 representation. Since $\nu_p \cap V_1$ and $\nu_p \cap V_2$ are G_p-invariant, this implies G_p is trivial on one of them, say, $\nu_p \cap V_1 =: \nu_p^1$. We can find $p_1' \in V_1$ in the normal slice at p_1, sufficiently close to p_1, such that $G_{p_1'} \subset G_{p_1}$ is a principal isotropy group of ρ_1. By replacing p by a G-conjugate, we may assume $G_{p_1'} = H_1$. Put $p' = p_1' + p_2$ and note that $G_{p'} = G_p$ and p' lies in the stratum of p, since G_p leaves ν_p^1 pointwise fixed. In particular, p' is a G-important point. Moreover

$$G_{p_1'+\lambda p_2} = (G_{p_1'})_{\lambda p_2} = (G_{p_1'})_{p_2} = G_{p'}$$

for all $\lambda \neq 0$, so $p_1' + \lambda p_2$ is also a G-important point, and hence $p_1' + 0$ projects to ∂X, by continuity. Then the slice representation of $G_{p_1'+0}$ has orbit space with non-empty boundary, but this representation equals the trivial action of H_1 on ν_p^1 plus $\rho_2|_{H_1}$. Hence $H_1 \neq \{1\}$ and $\partial Y_2 \neq \varnothing$. $\square$

We will need the following lemma communicated to us in much greater generality by the authors, see [21, Lemma 12.3]. Recall that a map between metric spaces is called a *submetry* if it maps any given closed ball around a point onto the closed ball of the same radius around the image point. For a connected complete Riemannian manifold M of positive curvature and closed subgroups $G \subset H$ of isometries of M, it is easy to see that the natural projection $M/G \to M/H$ is a submetry between Alexandrov spaces of positive curvature (on submetries, see also [4,27]).

Lemma 7.2 *(Kapovitch-Lytchak). For a compact Riemannian manifold M of positive curvature and closed subgroups of isometries $G \subset H$ of M, consider the natural submetry $f : X = M/G \to Y = M/H$. If $\partial X \neq \varnothing$ then $\partial Y \neq \varnothing$ (here we follow the usual convention that a point space has non-empty boundary).*

Proof. Suppose, to the contrary, that $\partial Y = \varnothing$. Since Y has no strata of codimension one, by [26, Lemma 4.1] we can find an infinite H-horizontal geodesic γ in M which meets no singular H-orbits and thus projects to a geodesic γ'' in the Alexandrov space Y.

Let γ' be the projection of γ to X. We can assume γ was chosen so that γ' starts at a point in $X \setminus \partial X$. Note that γ' is a horizontal lift of γ'' under f and hence a geodesic in the compact Alexandrov space X. By positive curvature of X, the distance function to ∂X is strictly concave and thus γ' must meet ∂X.

On the other hand, using [26, Lemma 4.1] again, we may assume γ was chosen so that γ' meets ∂X at a point x belonging to a codimension one stratum. Then γ' is a concatenation of geodesics that satisfies the reflection law at x, and hence cannot be locally minimizing at x, which is a contradiction. $\square$

Corollary 7.3. *Let $\rho : G \to \mathsf{O}(V)$ be a representation of a compact connected simple Lie group G with no trivial components and orbit space $X = V/G$. Assume $V = V_1 \oplus V_2$ is a G-invariant decomposition, write $\rho = \rho_1 \oplus \rho_2$, and put $X_1 = V_1/\rho_1(G)$ and $X_2 = V_2/\rho_2(G)$. If $\partial X \neq \varnothing$ then $\partial X_1 \neq \varnothing$ and $\partial X_2 \neq \varnothing$.*

Proof. Let H_1, H_2, Y_1 and Y_2 be as in Proposition 7.1; by this proposition, say $H_1 \neq \{1\}$ and $\partial Y_2 \neq \varnothing$. We claim that $H_1 \supsetneq \ker \rho_1$. In fact, otherwise $H_1 = \ker \rho_1 \supsetneq \{1\}$; this, together with the assumption that G is simple, yields that H_1 is a finite subgroup, but G is connected and no element of H_1 can act on V_2 as a reflection on a hyperplane, which is a contradiction to $\partial Y_2 \neq \varnothing$. Now it follows from the claim that $\partial X_1 \neq \varnothing$ [35, Lemma 3.1]. Finally, $\partial Y_2 \neq \varnothing$ is equivalent to $S(Y_2) = S(V_2)/\rho_2(H_1)$ having non-empty

Table 4
The invariant $\mathcal{L}_G$ for a compact connected simple Lie group G.

G	$\mathcal{L}_G$
$\mathsf{SU}(2)$	18
$\mathsf{SU}(n)$ $(n \geq 3)$	$2n^2 + 2n$
$\mathsf{SO}(n)$ $(n \geq 3)$	$n^2 + 3n$
$\mathsf{Sp}(3)$	48
$\mathsf{Sp}(4)$	72
$\mathsf{Sp}(5)$	102
$\mathsf{Sp}(n)$ $(n \geq 6)$	$4n^2$
G_2	36
F_4	96
E_6	132
E_7	222
E_8	396

boundary. The natural projection $S(Y_2) \to S(X_2)$ is a submetry, so Lemma 7.2 implies that $\partial S(X_2) \neq \varnothing$ and hence $\partial X_2 \neq \varnothing$. $\square$

8. Applications

8.1. Representations of simple groups

Proof of Theorem 1.2. We only sketch the main ideas. The main tools are Theorem 1.1, Lemma 3.1, Proposition 7.1 and Corollary 7.3. We list the related invariant $\mathcal{L}_G$ for the simple Lie groups G in Table 4. The full calculations are too long to reproduce here and we refer the reader to the unpublished manuscript [11] instead.

In order to obtain Table 1 (the irreducible case), for each simple group we bound the dimension of the candidate representations using Theorem 1.1. To exclude the representations whose dimensions fall within that bound but are not listed in Table 1, we check that they fail to satisfy another necessary condition for having non-empty boundary, namely, the existence of a nice involution (Lemma 3.1). We refer to the Borel-de Siebenthal classification of involutions of compact connected simple Lie groups [36, Theorem 8.10.8]. One can explicitly list the involutions and compute the codimensions of their fixed point sets to see that many involutions disobey the bound $4 + \dim G/K$ in the definition of nice involution. We also use some techniques from [12].

To see that the orbit space of $(\mathsf{Spin}(11), \mathbb{H}^{16})$ has non-empty boundary, note that the slice representation at a highest weight vector $(\mathsf{SU}(5), \mathbb{C}^5 \oplus \Lambda^2\mathbb{C}^5)$ (up to a trivial component of dimension 3) has non-empty boundary in the orbit space.

In order to obtain Table 2 (the reducible case), Corollary 7.3 says that we need only to check which sums of representations in Table 1 have orbit space with non-empty boundary. Here we can first apply the dimension estimate given by Theorem 1.1, and then proceed with the criterion given by Proposition 7.1. $\square$

8.2. Quaternionic representations

Proof of Corollary 1.5. It is equivalent to show that the tangent spaces of the $\hat{\rho}(G)$- and $\hat{\rho}(\mathsf{Sp}(1))$-orbits at a regular point of ρ meet in zero only. Therefore we may assume that G is a maximal closed connected subgroup of $\mathsf{Sp}(V)$. According to Dynkin, G is one of the following ($n = \dim_{\mathbb{H}} V$):

(i) $\mathsf{U}(n)$;
(ii) $\mathsf{Sp}(k) \times \mathsf{Sp}(n-k) \quad (1 \leq k < n)$;
(iii) $\mathsf{SO}(k) \otimes \mathsf{Sp}(n/k) \quad (3 \leq k \leq n)$;
(iv) a simple group.

We note that in cases (i) and (ii) the representation is reducible, contrary to our assumption.

In case (iii), the connected principal isotropy group of $\hat{\rho}$ is contained in G, which is sufficient. Indeed, the connected principal isotropy group of $\mathsf{SO}(k) \otimes \mathsf{Sp}(\ell)\mathsf{Sp}(1)$ is given by (cf. [16, p. 72]):

$$\begin{cases} \mathsf{SO}(k-4\ell) & \text{if } k \geq 4\ell+2; \\ \mathsf{Sp}(\ell-k) & \text{if } \ell \geq k+1; \\ \{1\} & \text{otherwise.} \end{cases}$$

In case (iv) we use Theorem 1.2. If the cohomogeneities of ρ and $\hat{\rho}$ do not differ by 3, the principal isotropy group $\hat{G}_{pr}$ of $\hat{\rho}$ is positive-dimensional. If, in addition, the principal isotropy group G_{pr} of ρ is non-trivial, then the orbit space of ρ has non-empty boundary [35, Lemma 3.1] and ρ must be listed in Table 1. Now ρ is one of:

$$(\mathsf{Spin}(11), \mathbb{H}^{16}),\ (\mathsf{Spin}(12), \mathbb{H}^{16}),\ (\mathsf{SU}(6), \Lambda^3\mathbb{C}^6),\ (\mathsf{Sp}(3), \Lambda_0^3\mathbb{C}^6),\ (\mathsf{E}_7, \mathbb{H}^{28}). \tag{8.8}$$

In the first representation we have a non-maximal group, as the half-spin representation of $\mathsf{Spin}(12)$ restricts to the spin representation of $\mathsf{Spin}(11)$. For the remaining four representations it is true that $c(\rho) = c(\hat{\rho}) + 3$ (see e.g. [18, Table A]).

There remains the case in which $\dim \hat{G}_{pr} > 0$ and G_{pr} is trivial. By the argument in Lemma 3.1 and Remark 3.2, there is a nice involution $\sigma \in \hat{G}_{pr}$ such that $\sigma^2 = 1$ and

$$\dim V - \dim V^\sigma \leq \dim \hat{G}/\hat{G}^\sigma. \tag{8.9}$$

Now $\sigma = (\sigma_1, \sigma_2) \in G \times \mathsf{Sp}(1)$, where $\sigma_2 = \pm 1$. Owing to the fact that G_{pr} is trivial, $\sigma_2 = -1$; further, $\sigma_1 \neq 1$ as σ is not central in $\hat{G}$. Now G acts almost effectively on the symmetric space of inner type $G/G^{\sigma_1} = \hat{G}/\hat{G}^\sigma$, and we can apply Proposition 4.2 as in section 5 to obtain that $\dim S(V) - \dim S(V)^G \leq \ell_{G/G^{\sigma_1}} \dim G/G^{\sigma_1} \leq \max_K\{\ell_{G/K} \dim G/K\} := \hat{\mathcal{L}}_G$, where K runs through all symmetric subgroups of G with

Table 5
The invariant $\hat{\mathcal{L}}_G$ for some compact simple Lie groups G.

G	$\hat{\mathcal{L}}_G$
$\mathsf{SU}(2)$	6
$\mathsf{SU}(n)$ $(n \geq 3)$	$2n^2 - 2n$
$\mathsf{SO}(n)$ $(n \geq 3)$	$n^2 - n$
$\mathsf{Sp}(n)$ $(3 \leq n \leq 6)$	$3n^2 + 3n$
$\mathsf{Sp}(n)$ $(n \geq 7)$	$4n^2 - 4n$
E_7	210

maximal rank, and $S(V)$ denotes the unit sphere of V. Due to the irreducibility of ρ, we have $S(V)^G = \varnothing$, so we deduce from this inequality that $\dim V \leq \hat{\mathcal{L}}_G$.

The compact simple Lie groups admitting irreducible representations of quaternionic type are listed in [16, p. 71], where also the minimal dimension of such a representation (of cohomogeneity at least 2) is given. In Table 5 we list the values of $\hat{\mathcal{L}}_G$ for those groups. Running through irreducible representations of quaternionic type of G of cohomogeneity at least 2 and dimension at most $\hat{\mathcal{L}}_G$, we precisely obtain those listed in (8.8) and $(\mathsf{Spin}(13), \mathbb{H}^{32})$.

We finally show that (8.9) cannot hold for the latter representation. Indeed in this case $V^\sigma = V^{-\sigma_1}$, and the calculation in [11, §2.2] shows that $\dim V^{-\sigma_1} = \frac{1}{2}\dim V = 64$, so V^σ has codimension 64, which is bigger than $\dim \hat{G}/\hat{G}^\sigma = \dim G/G^{\sigma_1} \leq \dim \mathsf{Spin}(13)/(\mathsf{Spin}(6) \times \mathsf{Spin}(7)) = 42$. □

We will use the following lemma in the proof of Corollary 1.4.

Lemma 8.1. *Let $\rho : G \to \mathsf{O}(V)$ be an irreducible representation of a compact Lie group of quaternionic type and cohomogeneity at least two. Assume $\tau : H \to \mathsf{O}(W)$ is a reduction of ρ. Then τ is also of quaternionic type.*

Proof. By assumption, the centralizer of $\rho(G)$ in $\mathsf{O}(V)$ contains an $\mathsf{Sp}(1)$-subgroup. Due to Corollary 1.5, this subgroup induces an $\mathsf{Sp}(1)$- or $\mathsf{SO}(3)$-group of isometries of $X := V/G = W/H$. By [28, Theorem A], any isometry in the identity component of the isometry group of X is induced by an element in the centralizer of $\tau(H)$ in $\mathsf{O}(W)$. We deduce that this centralizer has dimension at least 3. Since τ is irreducible [12, Lemma 5.1], this implies it is of quaternionic type. □

8.3. Representations of simple groups, continued

Proof of Corollary 1.4. A representation can admit a non-trivial reduction only if it has non-empty boundary in the orbit space. Therefore, in view of Table 1, it suffices to prove that the spin representation ρ of $G = \mathsf{Spin}(11)$ on $V = \mathbb{H}^{16}$ admits no non-trivial reductions. For later use, recall that its principal isotropy group is trivial and its cohomogeneity is 9.

Suppose, to the contrary, that ρ admits non-trivial reductions and choose a *minimal* reduction $\tau : H \to \mathsf{O}(W)$, that is, τ satisfies $W/H = V/G = X$, $\dim H < \dim G = 55$ and $\dim H$ is as small as possible. Then H_{pr} is trivial. Since ρ is of quaternionic type, by Lemma 8.1 also τ is of quaternionic type. In particular, H is semisimple. Since ρ is not toric [13], it also follows from [12, Theorem 1.7] that $\tau^0 = \tau|_{H^0}$ is irreducible.

Next, we need to analyze irreducible representations of quaternionic type (of dimension < 64) of compact connected semisimple Lie groups (of dimension < 55) of cohomogeneity 9.

Assume first H^0 is simple. It is easy to list irreducible representations of quaternionic type of simple groups of low dimension and estimate their cohomogeneities. This yields $H^0 = \mathsf{Sp}(1)$ and $W = \mathbb{H}^3$. In this case, W/H^0 has empty boundary, so H/H^0 is generated by elements that act on W/H^0 as reflections. It follows that there is an element $\sigma \in H\backslash H^0$ of order 2 fixing a H-important, H^0-regular point [12, §2.2, §4.3] and

$$\dim W - 1 = \dim H - \dim Z_H(\sigma) + \dim W^\sigma,$$

where $Z_H(\sigma)$ denotes the centralizer of σ in H, that is,

$$\dim W^\sigma = 8 + \dim Z_H(\sigma).$$

Note that $\dim Z_H(\sigma) = 1$ or 3 is odd. Due to [12, Lemma 11.1], $\dim W^\sigma$ is even, and we reach a contradiction.

We now assume H^0 is not simple. We can write $H^0 = H_1 \times H_2$, $W = W_1 \otimes_{\mathbb{R}} W_2$, $\tau = \tau_1 \otimes \tau_2$, where τ_1 is of real type and τ_2 is of quaternionic type. It follows from [12, Lemma 12.1] that the cohomogeneity

$$c(\mathsf{SO}(m) \otimes \mathsf{Sp}(n)) \geq c(\mathsf{SO}(3) \otimes \mathsf{Sp}(2)) \geq 3 \cdot 8 - (10 + 3) = 11$$

for $m \geq 3$ and $n \geq 2$ (see also [14, Lemma 3.5]), so we must have $\tau_2 = (\mathsf{Sp}(1), \mathbb{H})$. It follows that $\dim W_1 < 16$.

Let $p_i \in W_i$ be H_i-regular, for $i = 1, 2$. We estimate the cohomogeneity $c(\tau)$ by going to the slice at $p = p_1 \otimes p_2$, as follows. The normal space $\nu_p(H^0 p)$ decomposes as $\nu_{p_1}(H_1 p_1) \otimes \mathbb{R}p_2 \oplus (\nu_{p_1}(H_1 p_1) \ominus \mathbb{R}p_1) \otimes \mathbb{R}^3 \oplus T_{p_1}(H_1 p_1) \otimes \mathbb{R}^3$, and the connected H^0-isotropy at p has the form $(H_1)_{p_1} \times \{1\}$, acting thus trivially on the $\mathbb{R}^3$-factors and on the $\nu_{p_1}(H_1 p_1)$-factors. Therefore the cohomogeneity

$$\begin{aligned} c(\tau) &= c(\tau_1) + 3(c(\tau_1) - 1) + c((H_1)_{p_1}, 3(T_{p_1}(H_1 p_1))) \\ &\geq 4c(\tau_1) - 3 + c(\mathsf{SO}(m_1), 3\mathbb{R}^{m_1}) \quad (m_1 = \dim H_1 p_1) \\ &= 4c(\tau_1) + 3. \end{aligned}$$

Now $c(\tau) = 9$ implies $c(\tau_1) = 1$. From the classification of transitive linear actions on spheres, we deduce that τ_1 is one of

$$(\mathsf{SO}(n), \mathbb{R}^n),\ (\mathsf{G}_2, \mathbb{R}^7),\ (\mathsf{Spin}(7), \mathbb{R}^8),\ (\mathsf{Sp}(n)\mathsf{Sp}(1), \mathbb{R}^{4n});$$

the cohomogeneity of τ becomes, respectively, ≤ 7, ≥ 11, 8 and ≥ 16, a contradiction. This shows that a non-trivial reduction of ρ cannot exist. □

8.4. Isometric actions of certain simple groups

Lemma 8.2. *Let $M = G/K$ be a connected irreducible symmetric space, where G is the transvection group of M and K is connected. Assume M is not of Hermitian type. Consider the isotropy representation of K on the tangent space at the basepoint and denote by K_{pr} its principal isotropy group. Then $N_K(K_{pr})/K_{pr}$ is finite.*

Proof. Write $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ for the decomposition of the Lie algebra of G into the ±1-eigenspaces of the involution. There is a Cartan subspace $\mathfrak{a}$ of $\mathfrak{p}$ such that $K_{pr} = Z_K(\mathfrak{a})$. Let $k \in N_K(K_{pr})$. The action of k on $\mathfrak{p}$ must preserve the K_{pr}-isotypical decomposition of $\mathfrak{p}$. In particular, k stabilizes the K_{pr}-fixed point set in $\mathfrak{p}$. Since M is not of Hermitian type, the latter is $\mathfrak{a}$ [33, p. 11]. We get an inclusion $N_K(K_{pr}) \to N_K(\mathfrak{a})$ inducing an injective homomorphism (in fact, an isomorphism) $N_K(K_{pr})/K_{pr} \to N_K(\mathfrak{a})/Z_K(\mathfrak{a})$, where the target group is finite (it is the "little Weyl group" of M); this implies the desired result. □

Proof of Corollary 1.3. Suppose we are given a polar action of G on M. Then there are singular orbits [8, Lemma 2.1]. In particular we can find $p \in M$ and a positive dimensional isotropy group G_p. The slice representation at p is polar and has orbit space with non-empty boundary. It follows that p projects to the boundary of X.

Conversely, assume $\partial X \neq \varnothing$. Due to Theorem 1.1, M^G is non-empty and $\dim M^G \geq 1$; as in the proof of Theorem 1.6, it follows that any component of M^G of positive dimension is contained in ∂X. In particular, G has a fixed point $p \in M$ and the isotropy representation (G, T_pM) has orbit space with non-empty boundary. In case $G = \mathsf{SU}(2)$, Tables 1 and 2 say that $T_pM = \mathbb{C}^2$, up to a trivial representation. Now G acts transitively on the normal sphere to the component of M^G through p, so M is fixed point homogeneous and the result follows from [17, Classification Theorem 2.8].

In the other cases, Tables 1 and 2 say that the isotropy representation of G on T_pM is the isotropy representation of an irreducible symmetric space, not of Hermitian type, up to a trivial representation. It follows from Lemma 8.2 that the normalizer of the principal isotropy group G_{pr} is a finite extension thereof, which means the G-action on M is asystatic and, in particular, polar [10]. Now we can finish by using [8, Theorem A]. □

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